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Technical Note 1965-21

E. J. Kelly, Jr.

A Comparison of Seismic Array Processing Schemes

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

A COMPARISON OF SEISMIC ARRAY PROCESSING SCHEMES

E. J. KELLY, JR.

Group 64

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ABSTRACT

It is our purpose in this note to discuss three of the many approaches to seismic array processing from the theoretical point of view. The three are: 1) maximum-likelihood processing, 2) the minimum-variance, unbiased estimator (MVU) approach used by Levin, and 3) multichannel Wiener filtering. A feature common to these techniques is the formation of a single output waveform which serves as an estimator of the unknown signal coming from a fixed direction. We refer to such an output as a "beam."

It will be shown that 1) reduces to 2) in the case of gaussian noise and known signal parameters (i.e., known epicenter) and that 3) is related very simply to 2). In fact, the Wiener filtering output can be obtained directly from the MVU output. We treat both the sampled-data and continuous-time cases. The note contains no new results.

Accepted for the Air Force
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I. Introduction

Suppose that the array seismometers are identical (say short-period vertical instruments and are situated in a horizontal plane at the points $\underline{r}_k$, $k = 1, \ldots, N$. We write the output, $x_k(t)$, of the k^{th} seismometer as a sum of signal and noise:

$$x_k(t) = s_k(t) + n_k(t) \tag{1}$$

We suppose that the noise components have zero mean and are (wide-sense) stationary with covariance matrix

$$R_{k\ell}(t-t') = E\, n_k(t)\, n_\ell(t') \tag{2}$$

(Here E stands for expectation or ensemble average.) The signal is an unknown waveform, propagating across the array with horizontal phase velocity $\bar{v}$ and coming from a direction bearing β to the East of North. If this signal has the form S(t) at the origin of coordinates, it has the form $S(t-\underline{\alpha}\cdot\underline{r})$ at point $\underline{r}$, where

$$\alpha_x = \frac{\sin\beta}{\bar{v}}$$

$$\alpha_y = \frac{\cos\beta}{\bar{v}}$$

in a coordinate system where the x-axis points East and the y-axis points North. Thus,

$$s_k(t) = S(t-\underline{\alpha}\cdot\underline{r}_k) \tag{3}$$

We assume, at first, that the signal parameters (α_x and α_y, or $\bar{v}$ and β) are known, which is equivalent to knowing the epicenter if the depth of the event is assumed or known. Then the array can be "steered" by delaying the output of the k^{th} seismometer by the amount $\underline{\alpha}\cdot\underline{r}_k$. We call this output $x'_k(t)$:

$$x'_k(t) = x_k(t+\underline{\alpha}\cdot\underline{r}_k) = S(t) + n'_k(t) \tag{4}$$

where

$$n'_k(t) = n_k(t+\underline{\alpha}\cdot\underline{r}_k)$$

now has the covariance matrix

$$R'_{k\ell}(t-t') = E\, n'_k(t)\, n'_\ell(t') = R_{k\ell}[t-t' + \underline{\alpha}\cdot(\underline{r}_k - \underline{r}_\ell)] \tag{5}$$

The $x'_k(t)$ are, of course, equivalent to the $x_k(t)$ and no information has been destroyed.

II. The Minimum-Variance Unbiased Estimator

In this completely ad hoc approach we postulate a linear combination of the steered array outputs which is to provide a minimum variance unbiased estimate of the unknown signal S(t). Such an output beam has the form

$$y'(t) = \sum_{k=1}^{N} \int h_k(\tau)\, x_k'(t-\tau)\, d\tau \tag{6}$$

i.e., the steered array outputs are filtered and then summed. We do not impose realizability on the filter functions, $h_k(t)$, and all time integrals run from $-\infty$ to $+\infty$. Since

$$E\, x_k'(t) = S(t),$$

the expected output is

$$E\, y'(t) = \sum_{k=1}^{N} \int h_k(\tau)\, S(t-\tau)\, d\tau.$$

If $y'(t)$ is to be an unbiased estimator of S(t) for any S(t), we must have

$$\sum_{k=1}^{N} h_k(t) = \delta(t) \quad . \tag{7}$$

The variance of $y'(t)$, which is a constant since $y'(t)$ is a stationary process, is

$$\begin{aligned}\sigma^2 &= E\,[y'(t) - S(t)]^2 \\ &= E\,\Big\{ \sum_{k=1}^{N} \int h_k(\tau)\, n_k'(t-\tau)\, d\tau \Big\}^2 \\ &= \sum_{k=1}^{N} \iint h_k(\tau)\, h_\ell(\tau')\, R_{k\ell}'(\tau'-\tau)\, d\tau\, d\tau' \end{aligned} \tag{8}$$

The functions $h_k(t)$ are to be chosen to minimize expression (8) subject to constraint (7).

In the sampled-data version of the problem, we suppose that the steered array outputs are sampled every δ seconds, providing a sequence

$$\xi'_k(i) = x'_k(i\delta) \qquad\qquad i = 0, \pm 1, \pm 2, \text{ etc.}$$

and the output beam is the sequence

$$\eta(i) = \sum_{k=1}^{N} \sum_{j=-\nu}^{\nu} \theta_k(j)\, \xi'_k(i-j) \tag{9}$$

where it is supposed that the filters, $\theta_k(i)$, are of length $(2\nu+1)$ samples, symmetrically placed about zero (this last is non-essential assumption). Since

$$E\, \xi'_k(i) \equiv \sum(i) = S(i\delta),$$

the analogue of (7) is

$$\sum_{k=1}^{N} \theta_k(i) = \delta_{i,o} \quad . \tag{10}$$

In terms of the covariance matrix

$$P'_{k\ell}(i-j) = R'_{k\ell}(i\,\delta - j\,\delta) = P'_{\ell k}(j-i)\ ,$$

the output variance is just

$$\sigma^2 = E\, \eta(i)^2 = \sum_{k,\ell=1}^{N} \sum_{i,j=-\nu}^{\nu} P'_{k\ell}(j-i)\, \theta_k(i)\, \theta_\ell(j) \tag{11}$$

We solve the discrete version first, using the calculus of variations to minimize (11) subject to the set of $(2\nu+1)$ constraints (10). Using the symmetry of the covariance matrix, we find

$$\delta\sigma^2 = 2 \sum_{k,\ell=1}^{N} \sum_{i,j=-\nu}^{\nu} P'_{k\ell}(j-i)\, \delta\theta_k(i)\, \theta_\ell(j)\ ,$$

for the variation of σ^2 under variations, $\delta\theta_k(i)$, in the filter weights. We multiply the variation of the i^{th} constraint by the Lagrange multiplier $2\,\lambda(i)$, sum over i and add the result to $\delta\sigma^2$. The $\delta\,\theta_k(i)$ are now treated as independent variations, which leads immediately to the set of linear equations

$$\sum_{\ell=1}^{N} \sum_{j=-\nu}^{\nu} P'_{k\ell}(j-i)\,\theta_\ell(j) + \lambda(i) = 0 \tag{12}$$

where $k = 1,\ldots,N$ and $i = -\nu,\ldots,+\nu$.

Equations (12) could be solved for $\theta_k(i)$ in terms of the undetermined $\lambda(i)$, and the latter determined by substitution in the equations of constraint. However, since the constraints are also linear in the $\theta_k(i)$, we may adjoin equations (10) to equations (12), treat the $\lambda(i)$ as unknowns, and solve at once for the $\theta_k(i)$ and the $\lambda(i)$. We observe that (12) implies that σ^2 is equal to

$$\sigma^2 = \sum_{k=1}^{N} \sum_{i=-\nu}^{\nu} \theta_k(i)\,[-\lambda(i)]$$

$$= -\sum_{i=-\nu}^{\nu} \delta_{i,o}\,\lambda(i) = -\lambda(o)\,, \tag{13}$$

hence the value of $\lambda(o)$, at least, is of interest since it gives us the actual minimum noise variance attained. If we define

$$\theta_{N+1}(i) = \lambda(i) \tag{14}$$

we find that the system of equations can be written

$$\sum_{\ell=1}^{N+1} \sum_{j=-\nu}^{\nu} P'_{k\ell}(j-i)\,\theta_\ell(j) = \delta_{k,N+1}\,\delta_{j,o} \tag{15}$$

where we have extended the definition of the matrix P as follows:

$$P'_{k,N+1}(i-j) = P'_{N+1,k}(i-j) = \delta_{i,j} , \qquad k = 1,\ldots,N$$

and

$$P'_{N+1,N+1}(i-j) = 0.$$

The set of (N+1) (2ν+1) equations (15) is equivalent* to the set given by Levin in Reference (1).

In the continuous-time case, we could carry out a similar analysis in time domain, by introducing a Lagrangian function, $\lambda(t)$, to incorporate the infinity of constraints contained in (7), and leading to a system of integral equations for the functions $h_k(t)$ and $\lambda(t)$. However, if neither realizability nor finite memory is imposed on the $h_k(t)$, the solution is more easily obtained in frequency domain. We introduce the transform filter functions

$$H_k(\omega) = \int h_k(t)\, e^{i\omega t}\, dt = H_k^*(-\omega)$$

and the noise spectral density matrix of the steered outputs,

$$G'_{k\ell}(\omega) = \int R'_{k\ell}(t)\, e^{i\omega t}\, dt,$$

which has the symmetry properties

$$G'_{k\ell}(\omega) = G'_{\ell k}(-\omega) = G'^{*}_{\ell k}(\omega) .$$

In terms of the spectral density matrix, $G_{k\ell}(\omega)$, of the unsteered array outputs, $G'_{k\ell}(\omega)$ is given by the simple relation

$$G'_{k\ell}(\omega) = \exp\left[-i\omega\underline{\alpha}\cdot(\underline{r}_k - \underline{r}_\ell)\right] G_{k\ell}(\omega) \qquad (16)$$

The transform of the constraints (7) is simply

$$\sum_{k=1}^{N} H_k(\omega) = 1 \qquad \text{for all } \omega , \qquad (17)$$

---

* Levin's $h_k(i)$ equals our $\theta_k(-i)$ and his λ_i is our $\lambda(-i)$.

and expression (8) is equal to

$$\sigma^2 = \sum_{k,\ell=1}^{N} \int_{-\infty}^{\infty} G'_{k\ell}(\omega)\, H_k(\omega)\, H_\ell^*(\omega)\; d\omega/2\pi \tag{18}$$

It can be shown that $G'_{k\ell}(\omega)$ is a non-negative definite matrix for each ω, so that necessarily

$$\sum_{k,\ell=1}^{N} G'_{k\ell}(\omega)\, H_k(\omega)\, H_\ell^*(\omega) \geq 0 \;. \tag{19}$$

Hence (18) is minimized by minimizing the integrand, i.e., the left side of (19), at each frequency. Taking cognizance of the constraint (17) at each frequency, we obtain

$$H_k(\omega) = \frac{\sum_{\ell=1}^{N} Q'_{\ell k}(\omega)}{\sum_{m,n=1}^{N} Q'_{m,n}(\omega)} \tag{20}$$

where Q' is the inverse of the matrix G' at each frequency. The minimum noise variance obtained is given by

$$\sigma^2 = \int_{-\infty}^{\infty} \{ \sum_{k,\ell=1}^{N} Q'_{k\ell}(\omega) \}^{-1}\; d\omega/2\pi \tag{21}$$

From (20) we find that the $H_k(\omega)$ satisfy the equations

$$\sum_{k=1}^{N} G'_{k\ell}(\omega)\, H_k(\omega) = \{ \sum_{m,n=1}^{N} Q'_{m,n}(\omega) \}^{-1} \equiv \Lambda'(\omega) \tag{22}$$

for each ℓ. In time domain, this means that the $h_k(t)$ satisfy the system of integral equations

$$\sum_{\ell} \int R'_{k\ell}(t-s)\, h_k(s)\, ds = \lambda'(t) \tag{23}$$

where

$$\lambda'(t) = \int_{-\infty}^{\infty} \Lambda'(\omega)\, e^{-i\omega t}\, d\omega/2\pi \quad .$$

We also have, from (21),

$$\sigma^2 = \int_{-\infty}^{\infty} \Lambda'(\omega)\, d\omega/2\pi = \lambda'(0) \tag{24}$$

The function $-\lambda'(t)$ is just the Lagrangian multiplier function we mentioned above in a direct time-domain approach. Also, from (21) and (22) it is obvious that $\Lambda'(\omega)$ is the spectral density of the noise component of the output beam y(t).

III. The Maximum-Likelihood Processor

The maximum-likelihood form of processing has been discussed in detail in Reference 2. It is a constructive procedure for estimating both the signal waveform, S(t), and the signal parameters, $\underline{\alpha}$, considered to be unknown *a priori.* The noise is assumed to be gaussian and the estimation procedure takes place in two steps. First a tentative $\underline{\alpha}$ is chosen and an estimate, $\hat{S}(t;\underline{\alpha})$, of the signal waveform is constructed. This estimate is, in fact, the maximum-likelihood estimate of S(t) for the case in which $\underline{\alpha}$ is known and turns out to be exactly the beam, y'(t), discussed in II, i.e., a sum of the filtered, steered array outputs with filter functions given by (20).

In the general case, a modified likelihood function, $L(\underline{\alpha})$, is also constructed which is then maximized over $\underline{\alpha}$. The resulting value, $\hat{\underline{\alpha}}$, is the maximum-likelihood estimate of $\underline{\alpha}$ and $\hat{S}(t;\hat{\underline{\alpha}})$ is the final estimate of S(t). In terms of the transform, $Y'(\omega)$, of y'(t), the function $L(\underline{\alpha})$ is given by

$$L(\underline{\alpha}) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{|Y'(\omega)|^2}{\Lambda'(\omega)}\, d\omega/2\pi \tag{24}$$

The expression, a sort of signal-to-noise ratio, contains $\underline{\alpha}$ implicitly by way of the array steering indicated by the primes in our notation. In general, the final maximum-likelihood estimate of S(t) is biased, as would be any estimator for the case of unknown $\underline{\alpha}$ and more than one sensor. The accuracy of the estimate of $\underline{\alpha}$ is discussed in detail in Reference 2.

It can be shown that the maximum-likelihood filter weights for the estimation of signal with known parameters in the sampled-data case are also identical with those obtained by Levin.

IV. Multichannel Wiener Filtering

This case was also discussed briefly in Reference 2, but will be reviewed in greater detail here. As in II, we form a steered beam, y'(t), given by (6), but now we choose the filter functions to minimize the variance of the error function

$$\epsilon(t) = y'(t) - S(t) \quad , \tag{25}$$

where S(t) is taken to be a stationary random process with covariance function

$$R_o(t-t') = E\, S(t)\, S(t') = R_o(t'-t) \; ,$$

independent of the noise components.

First, we compute

$$E\, x_k'(t)\, x_\ell'(t') = R_o(t-t') + R_{k\ell}'(t-t')$$

and

$$E\, x_k'(t)\, S(t') = R_o(t-t') \quad ,$$

hence

$$E\, y'(t)^2 = \sum_{k,\ell=1}^{N} \iint h_k(\tau)\, h_\ell(\tau')\, E\, x'_k(t-\tau)\, x'_\ell(t-\tau')\, d\tau\, d\tau'$$

$$= \sum_{k,\ell=1}^{N} \iint h_k(\tau)\, h_\ell(\tau') \{R_o(\tau'-\tau) + R'_{k\ell}(\tau'-\tau)\}\, d\tau\, d\tau'$$

and

$$E\, y'(t)\, S(t) = \sum_{k=1}^{N} \int h_k(\tau)\, E\, x'_k(t-\tau)\, S(t)\, d\tau$$

$$= \sum_{k=1}^{N} \int h_k(\tau)\, R_o(-\tau)\, d\tau$$

$$= \sum_{k=1}^{N} \int h_k(\tau)\, R_o(\tau)\, d\tau \quad .$$

Finally, combining these results, we have

$$E\{y'(t) - S(t)\}^2 = \sum_{k,\ell=1}^{N} \iint h_k(\tau)\, h_\ell(\tau') \{R_o(\tau'-\tau) + R'_{k\ell}(\tau'-\tau)\}\, d\tau\, d\tau'$$

$$- 2 \sum_{k=1}^{N} \int h_k(\tau)\, R_o(\tau)\, d\tau + R_o(0)$$

At this point we reintroduce the Fourier transforms used in II and also the signal spectral density

$$G_o(\omega) = \int R_o(t)\, e^{i\omega t}\, dt \quad .$$

Expressing all time functions in terms of their Fourier transforms, we find

$$E\{y'(t) - S(t)\}^2 = \sum_{k,\ell=1}^{N} \int_{-\infty}^{\infty} H_k(\omega)\, H_\ell^*(\omega)\, \{G_o(\omega) + G'_{k\ell}(\omega)\}\, d\omega/2\pi$$

$$- \sum_{k=1}^{N} \int_{-\infty}^{\infty} \{H_k(\omega) + H_k^*(\omega)\}\; G_o(\omega)\; d\omega/2\pi$$

$$+ \int_{-\infty}^{\infty} G_o(\omega)\; d\omega/2\pi \;.$$

We now vary the $H_k(\omega)$ (as complex quantities) and find that the Wiener filter functions satisfy the equations

$$\sum_{k=1}^{N} \{G'_{k\ell}(\omega) + G_o(\omega)\}\; H_k(\omega) = G_o(\omega) \tag{26}$$

In terms of the inverse, $Q'_{k\ell}(\omega)$, to the matrix $G'_{k\ell}(\omega)$, we try a solution of the form

$$H_k(\omega) = \mu(\omega) \sum_{\ell=1}^{N} Q'_{\ell k}(\omega)\; G_o(\omega) \;.$$

We find that

$$\mu(\omega) = \{\, 1 + \frac{G_o(\omega)}{\Lambda(\omega)} \,\}^{-1}$$

and that the Wiener filter functions are

$$H_k(\omega) = \frac{\sum_{\ell=1}^{N} Q'_{\ell k}(\omega)}{\sum_{i,j=1}^{N} Q'_{ij}(\omega)} \cdot \frac{G_o(\omega)}{G_o(\omega) + \Lambda'(\omega)} \tag{27}$$

These functions are equal to the MVU filters, multiplied by the common filter function,

$$\frac{G_0(\omega)}{G_0(\omega) + \Lambda'(\omega)}$$

which is simply the appropriate infinite-memory, non-realizable Wiener filter to apply to the MVU output beam to convert it directly into a Wiener estimate of S(t).

The smoothing error committed by the multichannel Wiener filter is

$$E\{y'(t) - S(t)\}^2 = \int_{-\infty}^{\infty} G_0(\omega)\{1 - \sum_{k=1}^{N} H_k(\omega)\} \; d\omega/2\pi$$

$$= \int_{-\infty}^{\infty} \frac{G_0(\omega)\, \Lambda'(\omega)}{G_0(\omega) + \Lambda'(\omega)} \; d\omega/2\pi \tag{28}$$

The case of unknown $\underline{\alpha}$ is not directly amenable to Wiener smoothing methods.

The sampled data version of the Wiener filtering problem is easily formulated and formally solved.[3] However, because of the finite memory restrictions, we shall not find quite so direct a relationship to the sampled-data MVU solution as in the continuous time case.

We assume that the sampled signal sequence, $\sum(i)$, is statistically stationary with covariance matrix

$$E \sum(i) \sum(j) = P_0(i-j) = P_0(j-i) \, ,$$

and form a beam as in III:

$$\eta(i) = \sum_{k=1}^{N} \sum_{j=-\nu}^{\nu} \gamma_k(j) \; \xi'_k(i-j) \, . \tag{29}$$

We use $\gamma_k(i)$ for the filter weights to distinguish them from the MVU weights $\theta_k(i)$. We now require a minimum value for

$$E\,[\eta(i) - \sum(i)]^2 \quad ,$$

which leads immediately to the set of equations

$$\sum_{\ell=1}^{N} \sum_{j=-\nu}^{\nu} [P'_{k\ell}(j-i) + P_o(j-i)] \ \gamma_\ell(j) = P_o(i) \tag{30}$$

for $k = 1, \ldots, N$ and $i = -\nu, \ldots, \nu$.

In order to solve this system we modify the notation, and define the doubly-indexed matrices

$$\begin{aligned} A_{k,\ell}(i,j) &\equiv P'_{k,\ell}(j-i) \\ B_{k,\ell}(i,j) &\equiv P_o(j-i) \qquad\qquad k,\ell = 1,\ldots,N \end{aligned} \tag{31}$$

By $A^{-1}_{k,\ell}(i,j)$ we mean shall the inverse of A, in the sense that

$$\sum_{m=1}^{N} \sum_{s=-\nu}^{\nu} A_{k,m}(i,s)\, A^{-1}_{m,\ell}(s,j) = \delta_{k,\ell}\ \delta_{i,j} \tag{32}$$

Equations (30) now read

$$\sum_{\ell=1}^{N} \sum_{j=-\nu}^{\nu} [A_{k,\ell}(i,j) + B_{k,\ell}(i,j)] \ \gamma_\ell(j) = P_o(i) \ ,$$

and our problem is to invert the sum matrix A + B. We wish to express this inverse in terms of A^{-1} since A^{-1} appears in the MVU solution. We digress here to obtain expressions for the MVU filters in terms of A^{-1}. With our present notation, equation (12) reads

$$\sum_{\ell=1}^{N} \sum_{j=-\nu}^{\nu} A_{k,\ell}(i,j)\, \theta_\ell(j) = -\lambda(i) \ ,$$

hence

$$\theta_k(i) = -\sum_{\ell=1}^{N} \sum_{j=-\nu}^{\nu} A^{-1}_{k,\ell}(i,j)\, \lambda(j) \tag{33}$$

According to the constraints, (10), we have

$$\delta_{i,o} = \sum_{k=1}^{N} \theta_k(i) = - \sum_{j=-\nu}^{\nu} \sum_{k,\ell=1}^{N} A^{-1}_{k,\ell}(i,j)\, \lambda(j) \; .$$

We define the $(2\nu+1) \times (2\nu+1)$ matrix $a(i,j)$ by means of its inverse:

$$a^{-1}(i,j) \equiv \sum_{k,\ell=1}^{N} A^{-1}_{k,\ell}(i,j) \; . \tag{34}$$

Thus

$$\delta_{i,o} = - \sum_{j=-\nu}^{\nu} a^{-1}(i,j)\, \lambda(j)$$

and we have

$$\lambda(i) = - \sum_{j=-\nu}^{\nu} a(i,j)\, \delta_{j,o} = - a(i,o) \tag{35}$$

Finally, we have the desired expression for the MVU filters:

$$\theta_k(i) = \sum_{k=1}^{N} \sum_{j=-\nu}^{\nu} A^{-1}_{k,\ell}(i,j)\, a(j,o) \tag{36}$$

Had we asked that the MVU beam, $\eta(i)$, be an estimate of $\sum(i-s)$, for some integer s between $-\nu$ and $+\nu$, still using the $2\nu+1$ data prints symmetrically disposed about the point i, the analysis would be unchanged, except that in constraint (10) we have $\delta_{i,s}$ on the right side instead of $\delta_{i,o}$. We call the resulting MVU filters $\theta_k(i|s)$, and the multipliers that go with them $\lambda(i|s)$, and we have immediately

$$\theta_k(i|s) = \sum_{\ell=1}^{N} \sum_{j=-\nu}^{\nu} A^{-1}_{k,\ell}(i,j)\, a(j,s) \tag{37}$$

Of course, $\theta_k(i) = \theta_k(i|o)$ and $\lambda(i) = \lambda(i|o)$. We also find that the noise variance attained by the filters $\theta_k(i|s)$ is just $-\lambda(s|s) = a(s,s)$, while $a(s,s')$ = the covariance of the noise

outputs of the two processors $\theta_k(i|s)$ and $\theta_k(i|s')$. We have introduced these modified MVU filters because the Wiener filter weights are expressed in terms of a combination of all of them.

Returning to the Wiener filters, we make use of the matrix expansion analogue of

$$(1+x)^{-1} = \sum_{n=o}^{\infty} (-1)^n x^n ,$$

namely

$$(A+B)^{-1} = A^{-1} \sum_{n=o}^{\infty} (-1)^n (BA^{-1})^n . \tag{38}$$

[Here $(BA^{-1})^o$ is the unit matrix.]

Thus

$$\gamma_k(i) = \sum_{\ell,m=1}^{N} \sum_{j,s=-\nu}^{\nu} A^{-1}_{k,\ell}(i,j) \sum_{n=0}^{\infty} (-1)^n (BA^{-1})^n_{\ell,m}(j,s) P_o(s)$$

Now suppose we introduce the $(2\nu+1) \times (2\nu+1)$ matrix

$$b(i,j) = P_o(j-i) = B_{k\ell}(i,j) .$$

Then

$$\begin{aligned}
\sum_{\ell=1}^{N} \sum_{j=-\nu}^{\nu} (BA^{-1})_{k,\ell} (i,j) P_o(j) &= \sum_{\ell,m=1}^{N} \sum_{j,s=-\nu}^{\nu} B_{k\ell}(i,s) A^{-1}_{\ell m}(s,j) P_o(j) \\
&= \sum_{j,s=-\nu}^{\nu} b(i,s) \sum_{\ell,m=1}^{N} A^{-1}_{\ell m} (s,j) P_o(j) \\
&= \sum_{j,s=-\nu}^{\nu} b(i,s) a^{-1}(s,j) P_o(j) \\
&= \sum_{j=-\nu}^{\nu} [ba^{-1}] (i,j) P_o(j) \qquad , \quad k = 1,\ldots,N
\end{aligned}$$

in terms of the matrix product ba^{-1}. In fact,

$$\sum_{\ell=1}^{N} \sum_{j=-\nu}^{\nu} (BA^{-1})^{n}_{k,\ell}(i,j)\, P_o(j)$$

$$= \sum_{j=-\nu}^{\nu} [ba^{-1}]^{n}(i,j)\, P_o(j) \qquad , k = 1,\ldots,N ,$$

and therefore,

$$\gamma_k(i) = \sum_{\ell=1}^{N} \sum_{j=-\nu}^{\nu} A^{-1}_{k,\ell}(i,j) \sum_{n=o}^{\infty} (-1)^n [ba^{-1}]^n (j,s)\; P_o(j) \; .$$

But

$$\sum_{n=o}^{\infty} (-1)^n [ba^{-1}]^n = a(a+b)^{-1} ,$$

and therefore,

$$\gamma_k(i) = \sum_{\ell=1}^{N} \sum_{j,s,t=-\nu}^{\nu} A^{-1}_{k,\ell}(i,j)\, a(j,s)\, (a+b)^{-1}(s,t)\; P_o(j) \tag{39}$$

We define

$$W(i) \equiv \sum_{j=-\nu}^{\nu} (a+b)^{-1}\, (i,j)\; P_o(j)$$

and find the desired expression in terms of the MVU filters:

$$\gamma_k(i) = \sum_{j=-\nu}^{\nu} W(j)\; \theta_k(i|j) \tag{40}$$

which is analogous to equation (27) in the continuous case. It might be remarked that the sequence W(i) is vaguely related to the single-channel "signal/signal noise" Wiener filter. Finally, the Wiener smoothing error is

$$E\,[\eta(i) - \sum(i)\,]^2 = P_o(0) - \sum_{i,j=-\nu}^{\nu} (a+b)^{-1}(i,j)\,P_o(i)\,P_o(j) \tag{41}$$

We can also introduce a set of Wiener filter weights, $\gamma_k(i|s)$, in analogy to the MVU filters $\theta_k(i|s)$, by defining

$$\eta(i|s) = \sum_{k=1}^{N} \sum_{j=-\nu}^{\nu} \gamma_k(j|s)\;\xi_k'(i-j) \tag{42}$$

to be a Wiener estimate of $\sum(i-s)$. The only change is that $P_o(i)$ becomes $P_o(i-s)$ on the right side of (30) and $W(j)$ in (40) becomes $W(j|s)$, where

$$W(i|s) = \sum_{j=-\nu}^{\nu} (a+b)^{-1}(i,j)\;P_o(i-s)\;. \tag{43}$$

The corresponding smoothing error is

$$\begin{aligned} E\,[\eta(i|s) - \sum(i-s)\,]^2 &= P_o(0) - \sum_{i,j=-\nu}^{\nu} (a+b)^{-1}(i,j)\,P_o(i-s)\,P_o(j-s) \\ &= \{b - b(a+b)^{-1}\,b\}\,(s,s) \\ &= \{a\,(a+b)^{-1}\,b\}\,(s,s) \end{aligned} \tag{44}$$

Incidentally,

$$\begin{aligned} W(i|s) &= \sum_{j=-\nu}^{\nu} (a+b)^{-1}(i,j)\,b(j,s) \\ &= \{\,(a+b)^{-1}\,b\}\,(i,s)\quad. \end{aligned}$$

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13. ABSTRACT

It is our purpose in this note to discuss three of the many approaches to seismic array processing from the theoretical point of view. The three are: (1) maximum-likelihood processing, (2) the minimum-variance, unbiased estimator (MVU) approach used by Levin, and (3) multichannel Wiener filtering. A feature common to these techniques is the formation of a single output waveform which serves as an estimator of the unknown signal coming from a fixed direction. We refer to such an output as a "beam."

It will be shown that (1) reduces to (2) in the case of gaussian noise and known signal parameters (i.e., known epicenter) and that (3) is related very simply to (2). In fact, the Wiener filtering output can be obtained directly from the MVU output. We treat both the sampled-data and continuous-time cases. The note contains no new results.

14. KEY WORDS

seismometers	array processing	unbiased estimators
radar arrays	maximum-likelihood processing	Wiener filtering